COMBINATORICA 27 (3) (2007) 399–405
DOI: 10.1007/s00493-007-2105-z

COMBINATORICA
Bolyai Society – Springer-Verlag

NOTE

LOWER BOUND ON THE PROFILE OF DEGREE PAIRS IN CROSS-INTERSECTING SET SYSTEMS*

ZSUZSANNA SZANISZLÓ, ZSOLT TUZA

Received November 20, 2003

We raise the following problem. For natural numbers $m,n\ge 2$, determine pairs d',d'' (both depending on m and n only) with the property that in every pair of set systems $\mathcal{A},\mathcal{B}$ with $|A|\le m$, $|B|\le n$, and $A\cap B\ne\emptyset$ for all $A\in\mathcal{A},B\in\mathcal{B}$, there exists an element contained in at least $d'|\mathcal{A}|$ members of $\mathcal{A}$ and $d''|\mathcal{B}|$ members of $\mathcal{B}$. Generalizing a previous result of Kyureghyan, we prove that all the extremal pairs of (d',d'') lie on or above the line $(n-1)x+(m-1)y=1$. Constructions show that the pair $(\frac{1+\epsilon}{2n-2},\frac{1+\epsilon}{2m-2})$ is infeasible in general, for all $m,n\ge 2$ and all $\epsilon>0$. Moreover, for $m=2$, the pair $(d',d'')=(\frac{1}{n},\frac{1}{2})$ is feasible if and only if $2\le n\le 4$.

The problem originates from Razborov and Vereshchagin's work on decision tree complexity.

Preliminaries. Let $\mathcal{A}$ and $\mathcal{B}$ be two hypergraphs (set systems) on the same vertex set V. We say, for short, that $\mathcal{A}$ and $\mathcal{B}$ form an (m,n)-*crossing family* if

$|A|\le m$ for all $A\in\mathcal{A}$,
$|B|\le n$ for all $B\in\mathcal{B}$,
$A\cap B\ne\emptyset$ for all $A\in\mathcal{A}$ and all $B\in\mathcal{B}$.

Such families are interesting in various contexts. For instance, even if $\mathcal{A},\mathcal{B}$ are infinite, the requirement of cross-intersection is established on a suitably chosen subset $V'\subset V$ of *bounded size* in terms of m and n (a problem raised

Mathematics Subject Classification (2000): 05C65, 05C35

* Research supported in part by the Hungarian Research Fund under grant OTKA T-032969.

by Kuratowski, first studied by Całczyńska-Karłowicz [1]); and those families are also related to a class of Boolean functions that has been proposed as a model for the learning process of neurons, by Ehrenfeucht and Mycielski [2]. Tight asymptotic bounds have been given to these problems in [5]. For a detailed presentation of a related extremal method, we refer to the two-part survey [6,7].

Motivated by the study of *decision tree complexity*, Razborov and Vereshchagin [4] studied the problem of how large degrees must some vertices have *simultaneously* in $\mathcal{A}$ and $\mathcal{B}$, in *finite* (m,n)-crossing families. (The minterms and maxterms of a Boolean function – i.e., partial evaluations that make the function true or false, respectively – always intersect each other. Hence, if their sizes are bounded above by m and n, then an (m,n)-crossing family can be associated with them in a natural way.) To formulate explicit bounds, let us denote by $d_{\mathcal{A}}(v)$ and $d_{\mathcal{B}}(v)$ the number of sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ containing vertex v, respectively. Razborov and Vereshchagin proved by probabilistic methods that every finite (m,n)-crossing family contains a vertex v such that

$$d_{\mathcal{A}}(v) \geq \frac{|\mathcal{A}|}{2n} \qquad \text{and} \qquad d_{\mathcal{B}}(v) \geq \frac{|\mathcal{B}|}{2m}\,. \tag{1}$$

Moreover, they observed that (1) is tight apart from a multiplicative factor of 2, as shown in the collection of n disjoint m-tuples cross-intersected by m disjoint n-tuples.

Improved bounds. Kyureghyan [3] strengthened the lower bounds of (1), proving that there always exists a vertex v such that

$$d_{\mathcal{A}}(v) \geq \frac{|\mathcal{A}|}{2n-2} \qquad \text{and} \qquad d_{\mathcal{B}}(v) \geq \frac{|\mathcal{B}|}{2m-2}\,. \tag{2}$$

It is also shown in [3] that equality in (2) is attained for all $m,n \geq 2$.

The main contribution of our note is an extension of (2). We prove that if less is required for the degree of a vertex in $\mathcal{A}$ or in $\mathcal{B}$, then more can be ensured for its degree in the other set system.

Theorem 1. *Let $m,n \geq 2$ be integers. Let $p,q > 0$ be real numbers with $p+q=1$, $p \leq 1-\frac{1}{n}$ and $q \leq 1-\frac{1}{m}$. If the finite set systems $\mathcal{A},\mathcal{B}$ form an (m,n)-crossing family, then there exists a vertex v such that*

$$d_{\mathcal{A}}(v) \geq \frac{p\,|\mathcal{A}|}{n-1} \qquad \textit{and} \qquad d_{\mathcal{B}}(v) \geq \frac{q\,|\mathcal{B}|}{m-1}\,. \tag{3}$$

The proof given below not only yields a more general result than the one in [3], but also is simpler. Moreover, it can be strengthened further, cf. Remark 1 after the proof.

Currently we do not know what the strongest possible general lower bound in the style of (3) could be when we disregard the inequality conditions on p and q in the hypotheses of Theorem 1. Though this problem remains open even for $p=1/2$, at least it is true that for the value of $p=q$ one cannot take any real greater than $1/2$. On the other hand, some improvements for $m=2$ will be given later.

Tightness of (2). Before proving (1), we present an (m,n)-crossing family attaining equality in (2), whose structure is more transparent than the one described in [3].

Construction. If $m=2$ or $n=2$, then $2m-2=m$ or $2n-2=n$ holds, therefore Razborov and Vershchagin's construction (two disjoint sets of cardinality n or m, cross-intersected by n or m disjoint pairs) yields the required equality for the vertex degrees. (In fact, this leads to many non-isomorphic constructions, where $\mathcal{B}$ or $\mathcal{A}$ is *regular*.)

Suppose next that $m\geq 3$ and $n\geq 3$. Let P,Q,R,S and W,X,Y,Z be mutually disjoint nonempty sets whose cardinalities satisfy the conditions

$$|P|+|Q|=|Q|+|R|=|R|+|S|=|S|+|P|=n-1$$

and

$$|W|+|X|=|X|+|Y|=|Y|+|Z|=|Z|+|W|=m-1\,.$$

We define $\mathcal{A}$ as the collection of the following $2n-2$ sets of cardinality m:

$$\begin{gathered}\{p\}\cup W\cup X,\ \ p\in P\\ \{q\}\cup X\cup Y,\ \ q\in Q\\ \{r\}\cup Y\cup Z,\ \ r\in R\\ \{s\}\cup Z\cup W,\ \ s\in S\end{gathered}$$

Moreover, let $\mathcal{B}$ be the collection of the $2m-2$ n-element sets

$$\begin{gathered}\{w\}\cup Q\cup R,\ \ w\in W\\ \{x\}\cup R\cup S,\ \ x\in X\\ \{y\}\cup S\cup P,\ \ y\in Y\\ \{z\}\cup P\cup Q,\ \ z\in Z\end{gathered}$$

In this way we obtain an (m,n)-crossing family such that each $v\in P\cup Q\cup R\cup S$ is contained in precisely one $A\in\mathcal{A}$ and each $v\in W\cup X\cup Y\cup Z$ occurs in precisely one $B\in\mathcal{B}$. Thus, the degree condition holds with equality. ∎

Proof of Theorem 1. Given the (m,n)-crossing family $(\mathcal{A},\mathcal{B})$, let us consider an $|\mathcal{A}|\times|\mathcal{B}|$ matrix M where the rows and columns are labeled with the sets $A\in\mathcal{A}$ and $B\in\mathcal{B}$, respectively, and each entry $m_{A,B}$ is one element of $A\cap B$ (chosen arbitrarily if $|A\cap B|>1$). We say that an element $m_{A,B}$ is *frequent* in its row or column if it occurs at least $\frac{q|\mathcal{B}|}{m-1}$ times in the row labeled A or at least $\frac{p|\mathcal{A}|}{n-1}$ times in the column labeled B, respectively. Since no $A\in\mathcal{A}$ has more than m elements, $m\frac{q|\mathcal{B}|}{m-1}$ is a strict upper bound on the number of occurrences of the infrequent elements in row A. (Note that there are at most m such elements in the row.) Therefore the condition $\frac{qm}{m-1}\leq 1$ will guarantee the existence of at least one frequent element in row A. Of course the same argument applies to every row. However, this in turn means that fewer than $(m-1)\frac{q|\mathcal{B}|}{m-1}=(1-p)|\mathcal{B}|$ entries come from infrequent elements in every row. Hence more than $p|\mathcal{B}|$ entries correspond to frequent elements in every row. Similarly, more than $q|\mathcal{A}|$ entries correspond to frequent elements in every column. Therefore, there is an entry that is frequent both in its row and in its column. The corresponding vertex will satisfy the statement of the theorem. ∎

Remark 1. A statement slightly stronger than Theorem 1 is also true. Assuming that p is strictly smaller than $1-\frac{1}{n}$, under the conditions of Theorem 1 we can actually guarantee the existence of a vertex where the first inequality in (3) holds strict. To see this, notice in the proof above that in every row, still strictly fewer than $q|\mathcal{B}|$ entries correspond to infrequent elements; i.e., in the entire matrix the total number of elements frequent in their rows is definitely larger than $p|\mathcal{A}||\mathcal{B}|$. On the other hand, by the assumption $p<1-\frac{1}{n}$, at least $q|\mathcal{A}|$ elements are "strictly frequent" in each column, i.e. occur more than $\frac{p|\mathcal{A}|}{m-1}$ times in that column. Thus, by the pigeon-hole principle, at least one entry is frequent in its row and "strictly frequent" in its column.

Certainly, the analogous conclusion can be derived for the second inequality of (3) under the assumption $q<1-\frac{1}{m}$.

Best bounds for $m=2$ and n small. Next we investigate when one can guarantee the existence of a vertex of $\mathcal{A}$-degree $|\mathcal{A}|/n$ and $\mathcal{B}$-degree $|\mathcal{B}|/m$ at least. In fact this is the best one can expect, as shown by the m disjoint n-tuples and n disjoint m-tuples. By the tightness of (2) we also see that, in order to attain $|\mathcal{A}|/n$ and $|\mathcal{B}|/m$ simultaneously, we need to have $\min\{m,n\}=2$. The case of $m=n=2$ being trivial, the next results show that the necessary and sufficient condition is $\max\{m,n\}\leq 4$.

Below we assume $m=2$, i.e. that $\mathcal{A}$ is a graph. We begin with some simple observations.

Claim 1. *If $\mathcal{A}$ contains two adjacent vertices of $\mathcal{A}$-degree at least $|\mathcal{A}|/n$, then the degree pair $(|\mathcal{A}|/n, |\mathcal{B}|/2)$ is attained by at least one of these vertices.*

Proof. The claim is true since every $B \in \mathcal{B}$ meets every edge of $\mathcal{A}$. ∎

This settles all cases where $|\mathcal{A}| \leq n$.

Claim 2. *If $d_{\mathcal{A}}(v) \geq n$ for some vertex v, then the degree pair $(|\mathcal{A}|/n, |\mathcal{B}|/2)$ is attained at some vertex.*

Proof. In this case at most one $B \in \mathcal{B}$ can avoid v. If $|\mathcal{B}| = 1$, then the element of highest $\mathcal{A}$-degree in this B will do. Otherwise v has $\mathcal{B}$-degree at least $|\mathcal{B}| - 1 \geq |\mathcal{B}|/2$, and so does every other element with $\mathcal{A}$-degree $\geq n$. Then the vertex of largest $\mathcal{A}$-degree in any $B' \in \mathcal{B}$ containing v is a suitable choice. ∎

This settles all cases where $|\mathcal{A}| > n^2 - n$.

Theorem 2. *If $\mathcal{A}$ is 2-uniform and $\mathcal{B}$ is 3-uniform, then*

$$d_{\mathcal{A}}(v) \geq \frac{|\mathcal{A}|}{3} \qquad \text{and} \qquad d_{\mathcal{B}}(v) \geq \frac{|\mathcal{B}|}{2}$$

for some vertex v.

Proof. By the remarks above, we only need to consider the cases $4 \leq |\mathcal{A}| \leq 6$, with maximum $\mathcal{A}$-degree precisely 2, and with no pair of adjacent degree-2 vertices in $\mathcal{A}$. If $\mathcal{A}$ contains at least two vertices of degree 2, say v_1 and v_2, then the union of their disjoint neighborhoods is of size at least 4, i.e. every $B \in \mathcal{B}$ contains at least one of v_1 and v_2, hence yielding $\mathcal{B}$-degree at least $|\mathcal{B}|/2$. On the other hand, if v is the unique vertex of degree 2 in $\mathcal{A}$, then all the more than n edges are disjoint outside v and therefore all $B \in \mathcal{B}$ contain v. ∎

Theorem 3. *If $\mathcal{A}$ is 2-uniform and $\mathcal{B}$ is 4-uniform, then*

$$d_{\mathcal{A}}(v) \geq \frac{|\mathcal{A}|}{4} \qquad \text{and} \qquad d_{\mathcal{B}}(v) \geq \frac{|\mathcal{B}|}{2}$$

for some vertex v.

Proof. By the remarks above, we assume $5 \leq |\mathcal{A}| \leq 12$, and also that $\mathcal{A}$ has maximum degree at most 3 and no two vertices of $\mathcal{A}$-degree at least $|\mathcal{A}|/4$ are adjacent.

Case 1. $5 \le |\mathcal{A}| \le 8$ *and no two vertices of $\mathcal{A}$-degree 2 or 3 are adjacent.*

Then $\mathcal{A}$ is a vertex-disjoint union of stars. If just one of those stars has more than one edge, then its center vertex v has to be contained in every $B \in \mathcal{B}$ since the (at least 5) edges of $\mathcal{A}$ are disjoint outside v. In the other case we consider the centers v_1, v_2 of the first two nontrivial star components: $d_{\mathcal{A}}(v_1), d_{\mathcal{A}}(v_2) \ge 2$. It is now impossible to have a 4-element set that meets all edges of $\mathcal{A}$ outside $\{v_1, v_2\}$. Indeed, if $d_{\mathcal{A}}(v_1) + d_{\mathcal{A}}(v_2) \ge 5$ then already the edges incident with v_1 and v_2 would need more than 4 elements to cover; and if $d_{\mathcal{A}}(v_1) = d_{\mathcal{A}}(v_2) = 2$ then they would require just 4 elements but $\mathcal{A}$ still has at least one further edge to meet. In conclusion, every $B \in \mathcal{B}$ contains at least one of v_1 and v_2.

Case 2. $9 \le |\mathcal{A}| \le 12$ *and no two vertices of $\mathcal{A}$-degree 3 are adjacent.*

Since $\mathcal{A}, \mathcal{B}$ are cross-intersecting, every $B \in \mathcal{B}$ contains a vertex of $\mathcal{A}$-degree precisely 3. Let $d_{\mathcal{A}}(v_1) = 3$, and denote the neighbors of v_1 by u_1, u_2, u_3. All of them have degree at most 2. If some $B \in \mathcal{B}$ does not contain v_1, then $B \backslash \{u_1, u_2, u_3\}$ consists of a single vertex, say v_2. Then the only possibility to have $|\mathcal{A}| \ge 9$ is that $d_{\mathcal{A}}(u_i) = 2$ for all $1 \le i \le 3$ and v_2 is adjacent to 3 vertices outside $B \cup \{v_1\}$. But in this situation every member of $\mathcal{B}$ has to contain at least one of v_1 and v_2, since the 6 edges incident with $\{v_1, v_2\}$ are disjoint outside $\{v_1, v_2\}$. ∎

Remark 2. If $\mathcal{A}$ is 2-uniform and $\mathcal{B}$ is k-uniform, with $k > 4$, then a vertex v with

$$d_{\mathcal{A}}(v) \ge \frac{|\mathcal{A}|}{k} \qquad \text{and} \qquad d_{\mathcal{B}}(v) \ge \frac{|\mathcal{B}|}{2}$$

does not always exist. To see this for $k = 5$, let $\mathcal{A}$ be the union of three paths with two edges each. Let each $B \in \mathcal{B}$ consist of one vertex of degree 2 and the four degree-1 vertices not adjacent to that vertex. Then $|\mathcal{B}| = 3$ and $|\mathcal{A}| = 6$. For every vertex of degree 2 we have $d_{\mathcal{B}}(v) < \frac{|\mathcal{B}|}{2}$ and for every vertex of degree 1 we have $d_{\mathcal{A}}(v) < \frac{|\mathcal{A}|}{5}$. This construction easily generalizes for any larger k as well.

Open Questions. As we have already noted, the bounds of Theorem 1 are not sharp. Theorems 2 and 3 show the best possible bounds for $m = 2$ and $n = 3$ or 4. However, the same bound is not attainable for $n \ge 5$. What is the best possible bound when $m = 2$ and $n \ge 5$? And in general, what is the best possible bound for any m and n? In particular, for a given m, in the light of Theorem 1 it would be of interest to determine the largest possible value $d = d_m(n)$ such that every (m, n)-crossing family contains a vertex with $d_{\mathcal{A}}(v) \ge d|\mathcal{A}|$ and $d_{\mathcal{B}}(v) \ge \frac{|\mathcal{B}|}{2m-2}$.

References

[1] M. Całczyńska-Karłowicz: Theorem on families of finite sets, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **12** (1964), 87–89.
[2] A. Ehrenfeucht and J. Mycielski: Interpolation of functions over a measure space and conjectures about memory, *J. Approximation Theory* **9** (1973), 218–236.
[3] G. M. Kyureghyan: On a degree property of cross-intersecting families, *Combinatorica* **21(3)** (2001), 449–453.
[4] A. A. Razborov and N. K. Vereshchagin: A property of cross-intersecting families, in *Paul Erdős and his Mathematics*, Research Communications of the Conference held in the Memory of Paul Erdős, Budapest, Hungary, July 4–11, 1999 (A. Sali et al., eds.), J. Bolyai Math. Soc., Budapest, 1999, pp. 218–220.
[5] Zs. Tuza: On two intersecting set systems and k-continuous Boolean functions, *Discrete Applied Math.* **16** (1987), 183–185.
[6] Zs. Tuza: Applications of the set-pair method in extremal hypergraph theory, in *Extremal problems for finite sets* (P. Frankl et al., eds.), *Bolyai Society Mathematical Studies* **3**, J. Bolyai Math. Soc., Budapest, 1994, pp. 479–514.
[7] Zs. Tuza: Applications of the set-pair method in extremal problems, II; in: *Combinatorics, Paul Erdős is Eighty* (D. Miklós et al., eds.), *Bolyai Society Mathematical Studies* **2**, J. Bolyai Math. Soc., Budapest, 1996, pp. 459–490.

Zsuzsanna Szaniszló

Department of Mathematics
and Computer Science
Valparaiso University
Valpariso, IN 46383-6493
USA
Zsuzsanna.Szaniszlo@valpo.edu

Zsolt Tuza

Computer and Automation Institute
Hungarian Academy of Sciences
H-1111 Budapest
Kende u. 13–17.
and
Department of Computer Science
University of Veszprém
H-8200 Veszprém
Egyetem u. 10.
Hungary
tuza@sztaki.hu